JOURNAL OF APPROXIMATION THEORY 30, 155–156 (1980)

Note

On the Embedding Problem for Čebyšev Systems

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Communicated by Oved Shisha

Received May 18, 1979

It is shown by an example that there are continuously differentiable functions with only finitely many zeros that cannot be considered as elements of a Čebyšev system of continuous functions.

Let $M \subset \mathbb{R}$ and let $\mathbb{R}^M$ be the linear space of real-valued functions defined on M. An n-dimensional subspace U of $\mathbb{R}^M$ is called a Haar space and its bases are called Čebyšev systems if no $f \in U\setminus\{0\}$ has a strong alternation of length $n+1$ (see [2, Chap. 3]). Kurshan and Gopinath [1] showed that for an arbitrary $g \in \mathbb{R}^M$ there is a Haar space containing g if (and only if) g has only finitely many weak sign changes. They also raised the question whether this result holds for the continuous case, i.e., whether any $g \in C(M)$ with only finitely many weak sign changes can be embedded into a Haar space $U \subset C(M)$. We shall subsequently give an example showing that the answer is negative even if g is continuously differentiable.

Let $M = [0, \infty)$ and $g \in C(M)$ be defined by

$$\begin{aligned} g(x) &= 0 && \text{for } x = 0, \\ &= x^3\left(1 + \frac{x}{2} + \cos\left(\frac{\pi}{x}\right)\right) && \text{for } x > 0. \end{aligned}$$

Now suppose there exists an n-dimensional Haar space $U \subset C(M)$ with

0021-9045/80/100155-02$02.00/0

$g \in U$. Let $h \in U$ be a function with $h(0) = 1$, and $\delta > 0$ be chosen such that $\frac{1}{2} \leqslant h(t) \leqslant 2$ for $t \in [0, \delta]$.

Then one easily checks that for all k sufficiently large, one has

$$g\left(\frac{1}{k}\right)\Big/h\left(\frac{1}{k}\right) \leqslant \frac{1}{k^4} \qquad \text{for odd } k,$$

$$\geqslant \frac{1}{k^3} \qquad \text{for even } k.$$

So it is evident that for sufficiently small $\varepsilon > 0$, the function $g/h - \varepsilon^4$ has more than n zeros on $[\varepsilon^{4/3}, \varepsilon]$. The same then holds for $g - \varepsilon^4 h \in U$, a contradiction.

On the other hand, the answer is positive if one makes the additional hypothesis that g is strictly monotone in a neighbourhood of each of its zeros. This follows from a minor modification of the argument given in [2, Chap. 20].

References

1. R. P. Kurshan and B. Gopinath, Embedding an arbitrary function into a Tschebyscheff space, *J. Approx. Theory* **21** (1977), 126–142.
2. R. Zielke, "Discontinuous Čebyšev Systems," Lecture Notes in Mathematics No. 707, Springer-Verlag, Heidelberg, 1979.

Printed in Belgium